Dimensions of Crystalline Graded Rings

Tim Neijens
University of Antwerp
tim.neijens@gmail.com

Freddy Van Oystaeyen
University of Antwerp
fred.vanoystaeyen@ua.ac.be

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Abstract

The global dimension of a ring governs many useful abilities. For example, it is semi-simple if the global dimension is 0, hereditary if it is 1 and so on. We will calculate the global dimension of a Crystalline Graded Ring, as defined in the paper by E. Nauwelaerts and F. Van Oystaeyen, [10]. We will apply this to derive a condition for the Crystalline Graded Ring to be semiprime. In the last section, we give a little bit of attention to the Krull-dimension.

1 Preliminaries

Definition 1.1 ***Pre-Crystalline Graded Ring***
Let A be an associative ring with unit 1_A. Let G be an arbitrary group. Consider an injection $u : G \to A$ with $u_e = 1_A$, where e is the neutral element of G and $u_g \neq 0$, $\forall g \in G$. Let $R \subset A$ be an associative ring with $1_R = 1_A$. We consider the following properties:

(C1) $A = \bigoplus_{g \in G} Ru_g$.

(C2) *$\forall g \in G$, $Ru_g = u_g R$ and this is a free left R-module of rank 1.*

(C3) *The direct sum $A = \bigoplus_{g \in G} Ru_g$ turns A into a G-graded ring with $R = A_e$.*

*We call a ring A fulfilling these properties a **Pre-Crystalline Graded Ring**.*

Proposition 1.2 *With conventions and notation as in Definition 1.1:*

1. *For every $g \in G$, there is a set map $\sigma_g : R \to R$ defined by: $u_g r = \sigma_g(r) u_g$ for $r \in R$. The map σ_g is in fact a surjective ring morphism. Moreover, $\sigma_e = \mathrm{Id}_R$.*

2. *There is a set map $\alpha : G \times G \to R$ defined by $u_g u_h = \alpha(g,h) u_{gh}$ for $g, h \in G$. For any triple $g, h, t \in G$ the following equalities hold:*

$$\alpha(g,h)\alpha(gh,t) = \sigma_g(\alpha(h,t))\alpha(g,ht), \tag{1}$$
$$\sigma_g(\sigma_h(r))\alpha(g,h) = \alpha(g,h)\sigma_{gh}(r). \tag{2}$$

3. *$\forall g \in G$ we have the equalities $\alpha(g,e) = \alpha(e,g) = 1$ and $\alpha(g,g^{-1}) = \sigma_g(\alpha(g^{-1},g))$.*

Proof

See [10]. □

Proposition 1.3 *Notation as above, the following are equivalent:*

1. *R is $S(G)$-torsionfree.*
2. *A is $S(G)$-torsionfree.*
3. *$\alpha(g,g^{-1})r = 0$ for some $g \in G$ implies $r = 0$.*
4. *$\alpha(g,h)r = 0$ for some $g, h \in G$ implies $r = 0$.*
5. *$Ru_g = u_g R$ is also free as a right R-module with basis u_g for every $g \in G$.*
6. *for every $g \in G$, σ_g is bijective hence a ring automorphism of R.*

Proof

See [10]. □

Definition 1.4 *Any G-graded ring A with properties **(C1),(C2),(C3)**, and which is $G(S)$-torsionfree is called a **crystalline graded ring**. In case $\alpha(g,h) \in Z(R)$, or equivalently $\sigma_{gh} = \sigma_g \sigma_h$, for all $g, h \in G$, then we say that A is **centrally crystalline**.*

Lemma 1.5 *Let $R \underset{\sigma,\alpha}{\diamondsuit} G$ be a pre-crystalline graded ring, $x \in R$, $g, h \in G$. R is a domain, and define K to be the quotient field of R. Then*

1. $u_g^{-1} = u_{g^{-1}}\alpha^{-1}(x, x^{-1}) = \alpha^{-1}(x^{-1}, x)u_{x^{-1}}$.
2. $\sigma_g^{-1}(x)u_g^{-1} = u_g^{-1}x$.
3. $\sigma_{hg}^{-1}[\alpha(h,g)] = \sigma_g^{-1}[\sigma_h^{-1}(\alpha(h,g))]$.
4. $\sigma_g^{-1}[\alpha(g, g^{-1}h)] = \alpha^{-1}(g^{-1}, h)\sigma_g^{-1}[\alpha(g, g^{-1})]$.

Proof

(inverses are defined in K or $K \underset{\sigma,\alpha}{\diamondsuit} G$)

1. Just calculate the product and use that in an associative ring the left and right inverse coincide.

2. Let $g, h \in G, x \in A$:

$$\begin{aligned}
&\sigma_g[\sigma_h(x)]\alpha(g,h) = \alpha(g,h)\sigma_{gh}(x)\\
\Rightarrow&\sigma_g[\sigma_{g^{-1}}(x)]\alpha(g,g^{-1}) = \alpha(g,g^{-1})x\\
\Rightarrow&\sigma_{g^{-1}}(x)\sigma_g^{-1}(\alpha(g,g^{-1})) = \sigma_{g^{-1}}(\alpha(g,g^{-1}))\sigma_g^{-1}(x)\\
\Rightarrow&\sigma_g^{-1}(x) = \sigma_g^{-1}[\alpha^{-1}(g,g^{-1})]\sigma_{g^{-1}}(x)\sigma_g^{-1}[\alpha(g,g^{-1})].
\end{aligned}$$

So

$$\begin{aligned}
\sigma_g^{-1}(x)u_g^{-1} &= \sigma_g^{-1}[\alpha^{-1}(g,g^{-1})]\sigma_{g^{-1}}(x)\sigma_g^{-1}[\alpha(g,g^{-1})]\alpha^{-1}(g^{-1},g)u_{g^{-1}}\\
&= \sigma_g^{-1}[\alpha^{-1}(g,g^{-1})]\sigma_{g^{-1}}(x)\alpha(g^{-1},g)\alpha^{-1}(g^{-1},g)u_{g^{-1}}\\
&= \sigma_g^{-1}[\alpha^{-1}(g,g^{-1})]\sigma_{g^{-1}}(x)u_{g^{-1}}\\
&= \sigma_g^{-1}[\alpha^{-1}(g,g^{-1})]u_{g^{-1}}x\\
&= \alpha^{-1}(g,g^{-1})u_{g^{-1}}x\\
&= u_g^{-1}x.
\end{aligned}$$

3. Let $g, h \in G, x \in A$:

$$\begin{aligned}
&\sigma_h[\sigma_g(x)]\alpha(h,g) = \alpha(h,g)\sigma_{hg}(x)\\
\Rightarrow&\sigma_h[\sigma_g(\sigma_{hg}^{-1}(\alpha(h,g)))]\alpha(h,g) = \alpha(h,g)\sigma_{hg}(\sigma_{hg}^{-1}(\alpha(h,g)))\\
\Rightarrow&\sigma_{hg}^{-1}[\alpha(h,g)] = \sigma_g^{-1}[\sigma_h^{-1}(\alpha(h,g))].
\end{aligned}$$

4. Let $g, h \in G$:

$$\alpha(g,g^{-1})\alpha(e,h) = \sigma_g[\alpha(g^{-1},h)]\alpha(g,g^{-1}h).$$

$\square$

2 Global Dimension

Theorem 2.1 *Let R, S be rings with $R \subseteq S$ such that R is an R-bimodule direct summand of S, then* $\mathrm{r\ gld} R \leq \mathrm{r\ gld} S + \mathrm{pd} S_R$.
Proof See [7],p. 237. $\square$

Theorem 2.2 *Let R be a ring, G a finite group with $|G|$ a unit in R and $A = R \underset{\sigma,\alpha}{\diamondsuit} G$ a pre-crystalline graded ring with u_g units. Let M be any right A-module. Then:*

1. *If $N \lhd M_A$ and N is a direct summand of M as an R-module, then N is a direct summand over A.*
2. $\mathrm{pd} M_R = \mathrm{pd} M_A$.
3. $\mathrm{r\ gld} R = \mathrm{r\ gld} A$.

Proof

1. Let $\pi : M \to N$ be the R-module splitting morphism. Define the map λ by
$$\lambda : M \to N : m \mapsto |G|^{-1} \sum_{g \in G} \pi(m u_g) u_g^{-1}.$$
λ **is well-defined** : trivial.
λ **is the identity on** N : let $k \in N$:
$$\begin{aligned} \lambda(k) &= |G|^{-1} \sum_{g \in G} \pi(k u_g) u_g^{-1} \\ &= |G|^{-1} \sum_{g \in G} k = k. \end{aligned}$$
λ **is** A**-linear** : Let $m \in M, a \in A$:
$$\begin{aligned} \lambda(ma) &= |G|^{-1} \sum_{g \in G} \pi(m a u_g) u_g^{-1} \\ &= |G|^{-1} \sum_{g \in G} \pi \left[m \left(\sum_{h \in G} t_h u_h \right) u_g \right] u_g^{-1} \\ &= |G|^{-1} \sum_{g,h \in G} \pi \left(m t_h u_h u_g \right) u_g^{-1} \\ \text{(Lemma 1.5(2))} \quad &= |G|^{-1} \sum_{g,h \in G} \pi \left(m u_h u_g \right) u_g^{-1} \sigma_h^{-1}(t_h) \end{aligned}$$

$$
\begin{aligned}
&= |G|^{-1} \sum_{g,h\in G} \pi\left(m\alpha(h,g)u_{hg}\right) u_g^{-1}\sigma_h^{-1}(t_h) \\
&= |G|^{-1} \sum_{g,h\in G} \pi\left(mu_{hg}\right) \sigma_{hg}^{-1}[\alpha(h,g)]u_g^{-1}\sigma_h^{-1}(t_h) \\
\text{(Lemma 1.5(3))} \quad &= |G|^{-1} \sum_{g,h\in G} \pi\left(mu_{hg}\right) \sigma_g^{-1}[\sigma_h^{-1}(\alpha(h,g))]u_g^{-1}\sigma_h^{-1}(t_h) \\
\text{(Lemma 1.5(2))} \quad &= |G|^{-1} \sum_{g,h\in G} \pi\left(mu_{hg}\right) u_g^{-1}\sigma_h^{-1}[\alpha(h,g)]\sigma_h^{-1}(t_h) \\
(x=hg) \quad &= |G|^{-1} \sum_{h\in G}\sum_{x\in G} \pi\left(mu_x\right) u_{h^{-1}x}^{-1}\sigma_h^{-1}[\alpha(h,h^{-1}x)]\sigma_h^{-1}(t_h) \\
\text{(Lemma 1.5(4))} \quad &= |G|^{-1} \sum_{h\in G}\sum_{x\in G} \pi\left(mu_x\right) [\alpha^{-1}(h^{-1},x)u_{h^{-1}}u_x]^{-1}\cdot \\
&\qquad\qquad \alpha^{-1}(h^{-1},x)\sigma_h^{-1}[\alpha(h,h^{-1})]\sigma_h^{-1}(t_h) \\
&= |G|^{-1} \sum_{h\in G}\sum_{x\in G} \pi\left(mu_x\right) u_x^{-1}u_{h^{-1}}^{-1}\sigma_h^{-1}[\alpha(h,h^{-1})]\sigma_h^{-1}(t_h) \\
&= |G|^{-1} \sum_{h\in G}\sum_{x\in G} \pi(mu_x)u_x^{-1}u_h\sigma_h^{-1}(t_h) \\
&= |G|^{-1} \sum_{x\in G} \pi(mu_x)u_x^{-1} \sum_{h\in G} t_h u_h \\
&= \lambda(m)\cdot a.
\end{aligned}
$$

2. Suppose M_R is projective and

$$0 \to N \to F \to M \to 0$$

is a short exact sequence of A-modules with F free, then the sequence splits over R and hence over A by (1). So M_A is also projective. Furthermore, A_R is free. It now follows that an A-projective resolution of any module M_A is also an R-projective resolution that terminates when a kernel is, equally, R-projective or A-projective, so $\mathrm{pd}M_R = \mathrm{pd}M_A$.

3. Any A-module is naturally an R-module. So, since $\mathrm{pd}M_R = \mathrm{pd}M_A$, we find

$$
\begin{aligned}
\text{r gld}A &= \sup\{\mathrm{pd}M_A | M_A \text{ right } A-\text{module}\} \\
&\leq \sup\{\mathrm{pd}M_R | M_R \text{ right } R-\text{module}\} \\
&= \text{r gld}R.
\end{aligned}
$$

So by Theorem 2.1:

$$\begin{aligned} \text{r gld}R &\leq \text{r gld}A + \text{pd}A_R \\ &\overset{(2)}{=} \text{r gld}A + \text{pd}A_A \\ &= \text{r gld}A. \end{aligned}$$

And in conclusion r gld$R =$ r gld$A$. $\square$

The following result is well-known:

Lemma 2.3 *Let S be an Ore set for R and suppose there is no S-torsion. Let $\{s_1, \ldots, s_n\} \subset S$, then $\exists s \in S \cap \bigcap_{i=1}^n Rs_i$.*
Proof By induction. Let us take s_1, $\exists t_1 \in S^{-1}R$ such that $t_1 s_1 = 1$. Then of course we find $q_1 \in S$ such that $q_1 t_1 \in R$. This means that $q_1 = st_1 s_1 \in Rs_1$, and $q_1 \in S$. Now we try to do the same for the other s_i. We apply the left Ore condition on $q_1 \in S \subset R$ and $s_2 \in S$. We now find $v_2 \in R$ and $q_2 \in S$ such that $v_2 s_2 = q_2 q_1$. $\square$

Lemma 2.4 *Let $A = R \underset{\sigma,\alpha}{\diamondsuit} G$ be crystalline graded, then the set of regular elements in R, reg$R$, is a subset of reg$A$, the regular elements of A. Furthermore, if R is semiprime Goldie, reg$R$ is a left (and right) Ore set in A. We have*

$$(\text{reg}R)^{-1} A = \bigoplus_{g \in G} Q_{\text{cl}}(R) u_g.$$

Proof

For the first part, take $a \in \text{reg}R$, $x = \sum_{g \in G} x_g u_g$ and suppose $ax = 0$, then $\sum_{g \in G} a x_g u_g = 0$. This implies $a x_g = 0 \ \forall g \in G$, and this means $x_g, \forall g \in G$. Suppose $xa = 0$, then $\sum_{g \in G} x_g u_g a = 0$. This implies $x_g \sigma_g(a) u_g = 0$, or $x_g \sigma_g(a) = 0, \forall g \in G$. Since reg$R$ is invariant under $\sigma_g, \forall g \in G$, we again find $x_g = 0, \forall g \in G$. So we have proven reg$R \subset$ reg$A$.
By Goldie's Theorem, we know that reg$R$ is an Ore set in R. We first need to prove that $S =$ reg$R$ satisfies the left Ore condtion for A. We need that $\forall r \in R$, $s \in S$ we can find $r' \in R, s' \in S$ such that $s'r = r's$. Let $r = \sum_{g \in G} a_g u_g$. Since S is left Ore for R, we can find $\forall g \in G$ elements $a'_g \in R$ and $s_g \in S$ such that $a'_g \sigma_g(s) = s_g a_g$. Now, we find $s' \in S \cap \bigcap_{g \in G} Rs_g$ from Lemma 2.3, in other words, we find $s' \in S$ and $v_g \in R$ such that $\forall g \in G$ $s' = v_g s_g$. Now set $\forall g \in G$, $b_g = v_g a'_g$, and set $r' = \sum_{g \in G} b_g u_g$. Then $r's = s'r$. The right Ore condition is similar. The third assertion is now clear. $\square$

Theorem 2.5 *Let A be crystalline graded over R, R a semiprime Goldie ring. Assume* $\mathrm{char}R$ *does not divide $|G|$, then A is semiprime Goldie.*
Proof Since A is crystalline graded, the elements $\alpha(g,h), g,h \in G$ are regular elements. Denote $S = \mathrm{reg}R$. Since R is semiprime Goldie, $S^{-1}R$ is semisimple Artinian. This implies that from Theorem 2.2, $S^{-1}A$ is semisimple Artinian, in particular, it is Noetherian. Let I be an ideal in A, and consider $(S^{-1}A)I$. Claim: this is an ideal. Let $s \in S$ and consider the following chain:

$$(S^{-1}A)I \subset (S^{-1}A)Is^{-1} \subset (S^{-1}A)Is^{-2} \subset \ldots.$$

This implies that $(S^{-1}A)Is^{-n} = (S^{-1}A)Is^{-m}$, $m > n$, and so $(S^{-1}A)I = (S^{-1}A)Is^{n-m}$, and so we find $(S^{-1}A)I(S^{-1}A) \subset (S^{-1}A)I$, or $(S^{-1}A)I$ is an ideal in $S^{-1}A$. If J is the nilradical of A then $(S^{-1} \cdot J)^n = S^{-1} \cdot J^n$ follows. For some n we have that $(S^{-1} \cdot J)^n = 0$ in the semisimple Artinian ring $S^{-1}A$, thus $S^{-1}A \cdot J = 0$ and $J = 0$. $\square$

Corollary 2.6 *If A is crystalline graded with D a Dedekind domain,* $\mathrm{char}D$ *does not divide $|G|$, then A is semiprime.*

Proposition 2.7 *In the situation of Theorem 2.5, prime ideals of $S^{-1}A$ intersect in prime ideals of A, where $S = \mathrm{reg}R$.*
Proof Let P be a prime of $S^{-1}A$, then $P \cap Q$ is an ideal such that for $IJ \subset P \cap A$, I and J ideals of A, we have $S^{-1}A \cdot IJ \subset P$ hence $(S^{-1}A \cdot I)(S^{-1}A \cdot J) \subset P$, or $S^{-1}A \cdot I \in P$ if $S^{-1}A \cdot J \not\subset P$. Thus $I \subset P \cap A$ if $J \not\subset P \cap A$ and conversely. $\square$

Remark 2.8 *The situation of Theorem 2.5 arises when A is centrally crystalline graded over the semiprime Goldie ring R with* $\mathrm{char}R$ *does not divide $|G|$, such that A (or R) is a P.I. ring.*

3 Krull Dimension

Proposition 3.1 *Let A be crystalline graded over D, D a Dedekind domain. Then the (Krull-)dimension of A is smaller than or equal to* 2.
Proof Consider the set $F = \{I \triangleleft A | I \cap D = 0\}$ ordered by inclusion. If it is nonempty, then there is a maximal element for this family, say P. Suppose $IJ \subset P$, with $P \not\subset P + I$, $P \not\subset P + J$. Then $0 \neq d_1 \in P + I \cap D$ and

$0 \neq d_2 \in P + J \cap D$. This implies $0 \neq d_1 d_2 \in P$, contradiction. So if $F \neq \emptyset$, there always exists a prime ideal P in A with $P \cap D = 0$.
Denote $S = D \backslash \{0\}$. Suppose that $0 \neq Q \subset P$, Q a prime ideal in A. Then, since $S^{-1}A$ is Artinian semisimple (Theorem 2.2), we find that $S^{-1}Q = S^{-1}P$ since they are both primes ($Q \cap D \neq 0 \neq P \cap D$). Now let $y \in P \backslash Q$. Then $y \in S^{-1}P = S^{-1}Q$. This means $\exists d \in S$ such that $dy \in Q$. So if we set $d' = \prod_{g \in G} \sigma_g(d)$ then $d'y \in Q$. Since $d' \in Z(A)$ we find $d'Ay \subset Q$ and since $y \notin Q$ we see that $d' \in Q$ or $Q \cap D \neq 0$. Contradiction. We have established that two prime ideals that don't intersect D cannot contain each other.
Suppose there exists a prime ideal M of A with $M \cap D \neq 0$. This means A/M is Artinian, and prime, in other words it is a simple ring, or M is a maximal ideal. We find that a maximal chain of prime ideals always is of the form

$$0 \subset P \subset M \subset A,$$

where $P \cap D = 0$ and $Q \cap D \neq 0$. $\square$

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